Modal feedback control on chaotic trajectories

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The definition of Lyapunov exponents over a finite time interval is reviewed. It is shown that a bounded, invertible coordinate transformation exists on the tangent space which diagonalizes the uncontrolled system, thus introducing *modal variables* for the system. An algorithm is given which allows a Lyapunov exponent to be deterministically changed to any other real value (at least over a finite trajectory arc), while leaving all other Lyapunov exponents unchanged. This cannot be done in general with a constant gain, but can be done if the gain is a function of time. Iteration of this algorithm allows for alteration of any subset of the Lyapunov exponent spectrum. A numerical example based on the Lorenz system is given.

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I. INTRODUCTION

Recently there has been a great interest in control of chaotic trajectories. There are three main directions to this work. Much of this work follows the suggestion by Ott, Grebogi, and Yorke [1] that the parameters of a system can be used for controlling the evolution of the trajectory. An excellent recent review is given by Shinbrot, Grebogi, Ott, and Yorke [2]. Controlling the system to a set of goal dynamics was pioneered by Lüscher and Hübler [3], among others. Finally, application of feedback control to chaotic problems is still in its infancy. Pyragas [4] has applied feedback control with gains determined through numerical experiments, while Romeiras *et al.* [5] have applied feedback control on a surface of section, where the well-known constant coefficient control methods are applicable. Usually these efforts stabilize one of the unstable periodic orbits embedded within a chaotic attractor, so the control problem becomes one of stabilizing an unstable periodic orbit. We note that the problem of control of unstable periodic orbits was solved using linear quadratic regulators by Breakwell, Kamel, and Ratner [6], and the pole placement problem for periodic orbits was solved by Calico and Wiesel [7].

Situations exist where it is desired to use a chaotic trajectory, not a periodic orbit, and system parameter control is not applicable. For example, many interplanetary probes utilize chaotic, multiflyby trajectories which are very sensitive to their initial conditions, yet these trajectories are almost totally insensitive to the changes in system parameters (planetary masses and orbits) which are within the reach of human engineers. Rather, these orbits are controlled by maneuvering the spacecraft, using the sensitivity of the chaotic trajectory to greatly amplify the effects of a small velocity change, as far into the future as can be deterministically predicted. Also, usually these spacecraft are launched into the correct trajectory, as closely as possible, so producing dramatic shifts between trajectories is not required. The other, and perhaps more common, approach to control is *feedback control*, in which the scientist or engineer has direct (although perhaps limited) ability to change the system equations of motion. The control terms usually appear as additive functions in the equations of motion, whose character must be chosen to stabilize the desired trajectory. It is feedback control of chaotic systems through the system equations of motion that is the subject of this paper. In many ways this paper is similar in approach to Calico and Wiesel [7], whose authors gave the first modal feedback algorithms for time-periodic control problems. We differ from Pyragas [4] in that we will attempt to directly alter the Lyapunov exponents of the trajectory by predetermined amounts. We differ from Romeiras *et al.* [5] in that we will apply control throughout the trajectory.

A general dynamical system can be written as a vector set of differential equations

$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X}, t), \tag{1}$$

where $\mathbf{X}$ is termed the state vector. Introduce the small displacement $\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}_0(t)$ from a known trajectory $\mathbf{X}_0(t)$. Then, to first order in small quantities, the displacement vector obeys the variational equations

$$\dot{\mathbf{x}} = A(t)\mathbf{x} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{X}}\right|_{\mathbf{X}_0} \mathbf{x}. \tag{2}$$

As a set of linear equations, the variational equations are formally solved by the fundamental matrix $\Phi(t, t_0)$, which obeys

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(t_0, t_0) = I. \tag{3}$$

Then, the general solution to (2) can be written as $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}(t_0)$.

II. REGIONAL LYAPUNOV EXPONENTS

The stability of a general trajectory of a linear system is determined by the Lyapunov exponents. These are the values

$$\lambda_i = \frac{1}{t_f - t_0} \ln \frac{|\Phi(t_f, t_0)\mathbf{e}_i(t_0)|}{|\mathbf{e}_i(t_0)|}, \tag{4}$$

extremalized over all initial displacements $\mathbf{e}_i(t_0)$. Usually, the above includes a limit as $t_f \to \infty$, but in this work we are interested in finite times. This restriction to finite time intervals is an absolute necessity, since the idea of control implies that the future can be predicted so that action can be taken to influence its outcome. This is only possible with chaotic trajectories for a finite time interval.

The vectors $\mathbf{e}_i(t_0)$ represent extrema in the growth rate of the norm of displacement vectors, $|\mathbf{x}(t_f)|$ with respect to the initial displacement $\mathbf{x}(t_0)$. This cannot be an unconstrained maximization, since using an initial displacement $\mathbf{x}(t_0)$ with a larger magnitude will, of course, increase the final result. So, constrain $|\mathbf{x}(t_0)|$ to be unity. We then have the constrained optimization problem

$$J = |\mathbf{x}(t_f)|^2 - \mu \left(|\mathbf{x}(t_0)|^2 - 1 \right), \tag{5}$$

which must be extremalized with respect to the components of $\mathbf{x}(t_0)$. In the above, μ is a Lagrange multiplier. Now, since $\mathbf{x}(t_f) = \Phi(t_f, t_0)\mathbf{x}(t_0)$, the scalar function (5) becomes

$$J = \sum_i \left\{ \sum_j \Phi_{ij} x_{0j} \right\}^2 - \mu \left\{ \sum_i x_{0i}^2 - 1 \right\} \tag{6}$$

in terms of the components x_{0i} of the initial displacement vector. Partial derivatives can now be calculated as if all components were independent, yielding

$$\frac{1}{2} \frac{\partial J}{\partial x_{0k}} = 0 = \sum_i \sum_j \Phi_{ij} \Phi_{ik} x_{0j} - \mu_i x_{0k}, \tag{7}$$

where $k = 1, 2, \ldots, N$.

The solutions to (7) with unit magnitude are the desired $\mathbf{e} = \mathbf{x}(t_0)$ vectors at time t_0. But (7) can be recognized as the component form of

$$\left\{ \Phi^T \Phi - \mu_i I \right\} \mathbf{e}_i(t_0) = 0. \tag{8}$$

That is, the $\mathbf{e}_i(t_0)$ are the real, orthogonal eigenvectors of the real symmetric matrix $\Phi^T \Phi$. Comparison to (4) shows that the Lyapunov exponents over the time interval $(t_0,\ t_f)$ are just

$$\mu_i = \exp\left\{ 2\lambda_i (t_f - t_0) \right\}. \tag{9}$$

This has been recognized by Goldhirsch, Sulem, and Orszag [8]. In the limiting case where $\delta t = t_f - t_0$ becomes small, the fundamental matrix becomes $\Phi(t, t_0) \approx I + A\delta t$. Using this in (8) gives

$$\left\{ \left(A + A^T\right) \delta t - (\mu_i - 1) I \right\} \mathbf{e}_i = 0. \tag{10}$$

This is the short term limit for the local extremal expansion and/or contraction directions. Expanding (9) for small δt gives $\mu_i - 1 \approx 2\lambda_i \delta t$, so locally the Lyapunov exponents become the eigenvalues of $(A + A^T)/2$. In the long term limit as $t \to \infty$, we obtain the Lyapunov exponents of the chaotic trajectory. However, in this work we will focus on intermediate times. Since we are not interested in either the infinitesimal limit, leading to local Lyapunov exponents, nor the infinite time limit, leading to the classical Lyapunov exponents, we will refer to our λ_i as *regional* Lyapunov exponents, over the finite time interval $(t_0,\ t_f)$.

III. LOCAL DECOUPLING

Over a finite arc of the trajectory, the regional Lyapunov exponents may be used to factor the dynamics into separate modes. The initial conditions $\mathbf{e}_i(t_0)$ introduce N special solutions to the variational equations, $\mathbf{x}_i(t) = \Phi(t, t_0)\mathbf{e}_i(t_0)$, on which the average exponential rate of expansion or contraction is an extremum. But local variations in these rates can be quite large; see Haubs and Haken [9], Nese [10], Sepúlveda, Badii, and Pollak [11]. We wish to use these N special solutions to the variational equations as basis vectors for the entire solution set, and it would be very inconvenient for them to be anything other than unit vectors. Their instantaneous rate of change of magnitude is given by

$$\sigma_i(t) = \frac{\mathbf{x}_i \cdot A\mathbf{x}_i}{|\mathbf{x}_i|^2}. \tag{11}$$

Since the regional Lyapunov exponents are the average of these instantaneous rates on these N extremal solutions, we have

$$\lambda_i = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} \sigma_i(\tau) d\tau. \tag{12}$$

Then, define N new functions $\mathbf{e}_i(t)$ as the solutions to

$$\dot{\mathbf{e}}_i(t) = A\mathbf{e}_i - \sigma_i(t)\mathbf{e}_i \tag{13}$$

with initial conditions $\mathbf{e}_i(t_0)$ on the interval (t_0, t_f). Since the particular solution to the variational equations $\mathbf{x}_i = \Phi\mathbf{e}_i(t_0)$ grows exponentially at the average rate of λ_i over the interval $(t_0,\ t_f)$, we note that the scaled solution $\mathbf{e}_i(t)$ above does *not* grow exponentially over the defining time interval, either instantaneously or in the average. By construction, the $\mathbf{e}_i$ vectors are orthonormal at $t = t_0$. They are, by (13), trivially unit vectors on the entire interval $t_0 \le t \le t_f$. That they must also be orthogonal at $t = t_f$ can be seen by realizing that $\mathbf{e}_i(t_f)$ must also be the extremal initial conditions for exponential growth of trajectories running *backwards* in time. So they are the eigenvectors of the symmetric matrix $(\Phi^{-1})^T(\Phi^{-1})$, and are orthogonal. But at other times in the interval $(t_0,\ t_f)$ the $\mathbf{e}_i(t)$ vectors may not be orthogonal. We note that since the new vectors remain unit vectors,

$$\sigma_i(t) = \mathbf{e}_i \cdot A\mathbf{e}_i \tag{14}$$

is an alternate form of (11). The $\mathbf{e}_i$ have the same direction as the special solutions $\mathbf{x}_i$ throughout the time interval, differing from them only in magnitude.

Now, assemble the $\mathbf{e}_i(t)$ vectors by columns into the matrix $\mathcal{E}(t)$. The matrix analog of (13) is easily found to be

$$\dot{\mathcal{E}} = A\mathcal{E} - \mathcal{E}J(t), \tag{15}$$

where $J(t)$ is the diagonal matrix whose entries are the $\sigma_i(t)$. This is a relationship which is very familiar from time-periodic systems.

We wish to use the $\mathbf{e}_i(t)$ vectors as the coordinate vectors for describing the solution to the variational equations. To this end, define new coordinates $\mathbf{y}$ on the tangent space as

$$\mathbf{x}(t) = \mathcal{E}(t)\mathbf{y}(t). \tag{16}$$

Since $\mathcal{E}(t)$ is a nonsingular matrix function of time, at least for $t_0 \leq t \leq t_f$, all stability information resides within the $\mathbf{y}$ variables. [We know that the $\mathcal{E}$ matrix is nonsingular, since the $\mathbf{x}_i(t)$ are assumed distinct on our finite time interval.] Again differentiating (16) and substituting into the variational equations (2) we have

$$\dot{\mathbf{y}} = \left\{\mathcal{E}^{-1}A\mathcal{E} - \mathcal{E}^{-1}\dot{\mathcal{E}}\right\}\mathbf{y}. \tag{17}$$

But using (15), this easily reduces to

$$\dot{\mathbf{y}} = J(t)\mathbf{y}. \tag{18}$$

So, this transformation takes the variational equations (2), and replaces them with a set of *decoupled, time-dependent coefficient* differential equations for the variables $\mathbf{y}$, and another set of linear equations (15) for the coordinate vectors $\mathbf{e}(t)$. We will refer to $\mathbf{y}$ as the *modal variables* for the system, and $\mathcal{E}(t)$ as the modal matrix. This is the general case of the modal transformation implicit in Floquet theory for time-periodic systems, and used by Calico and Wiesel for feedback control of periodic orbits.

[We note here parenthetically that the choice $\sigma_i = \lambda_i$ also leads to another viable decoupling. Equations (15), (16), and (18) are unchanged, except that J is now the diagonal matrix of the constant regional Lyapunov exponents, so (18) represents a set of *constant coefficient* differential equations. However, the variation in the local Lyapunov exponents makes this choice, in the opinion of the author, far less desirable.]

We have applied this technique to the Lorenz attractor, [12], as modified by Shinbrot, Ott, Grebogi, and Yorke [13] to include a control input. The system of differential equations is

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= -xz + \gamma x - y + u(t), \\ \dot{z} &= xy - bz. \end{aligned} \tag{19}$$

The parameter values we have used are $\sigma = 16$, $\gamma = 40$, $b = 4$, while $u(t)$ is the control input. For $u(t) = 0$ this system has the well known "butterfly" attractor. For all the cases considered in this paper, the base line trajectory begins at $x = 2.426\,881\,355\,528$, $y = 2.577\,259\,040\,064$, $z = 26.689\,700\,667\,84$. This point on the attractor was found by a long numerical integration of an arbitrary set of initial conditions. The Lyapunov exponents for this system as $t \to \infty$ have been calculated by Shimada and Nagashima [14], and are $\lambda_{i,\infty} = 1.37,\ 0,\ -22.37$.

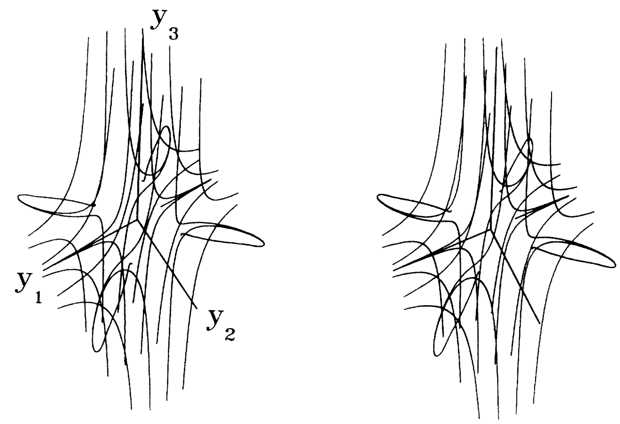

FIG. 1. The tangent space of the reference trajectory in modal variables. The plotted region spans $|y_i| \leq 0.001$ distance units in the modal space. At the core of the figure a perfect saddle point appears, while trajectories at the edge exhibit nonlinear behavior.

Decoupling begins by integrating the system state and variational equations in parallel, and examining the eigenvalues and vectors of $\Phi^T\Phi$. For this system, the complete spectrum of regional Lyapunov exponents is only well determined for time intervals $t_f - t_0 \approx 0.5$ time units, using double precision arithmetic. This is due to the fact that the smallest exponent, according to (9), is decreasing like $\mu_3 \approx \exp[-44(t_f - t_0)]$, and the eigenvalue package cannot accurately compute it for large final times. However, at $t_f - t_0 = 0.5$ we believe we have an accurate set of regional exponents, with values $\lambda_i = 2.0489,\ -0.6365,\ -22.4123$. This trajectory arc is long enough that the regional exponents are beginning to approach their final values. The propagation of the modal matrix over this interval using (15) was then performed. A strong check is that the final modal matrix $\mathcal{E}(t_f)$ is very accurately orthonormal. This check is also reminiscent of the periodic case, where the modal matrix must be periodic.

With the modal matrix constructed, trajectories on the tangent space can be examined. This was done by integrating the reference trajectory $\mathbf{X}_0(t)$, a nearby trajectory $\mathbf{X}(t)$, and the modal matrix $\mathcal{E}$ all in parallel. Then local modal variables were calculated from

$$\mathbf{y} = \mathcal{E}^{-1}\mathbf{x} \approx \mathcal{E}^{-1}\left[\mathbf{X}(t) - \mathbf{X}_0(t)\right]. \tag{20}$$

The results are shown in Fig. 1 as a stereo pair. The y_1 axis is the unstable manifold of the system, while the y_2 and y_3 axes are stable. The success of the modal transformation is shown by the appearance of a perfect saddle point near the origin in modal coordinates. The local character of this transformation is emphasized by the behavior of trajectories near the boundary of the plotted region, which is limited to $|y_i| \leq 0.001$ dimensionless units.

IV. MODAL FEEDBACK CONTROL

A linear dynamical system which is subject to a control system is usually written as

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \tag{21}$$

where $\mathbf{u}$ is the vector of control variables, and the matrix B apportions the control to the physical states which can be influenced. For example, in a particle dynamics problem, only the momentum states can be directly influenced by applying control forces. The coordinates are not subject to direct control. Introducing the modal variables, (21) becomes

$$\begin{aligned}\dot{\mathbf{y}} &= J\mathbf{y} + \mathcal{E}^{-1}B\mathbf{u}, \\ &= \left\{J + \mathcal{E}^{-1}BG\right\}\mathbf{y},\end{aligned} \tag{22}$$

the second line assuming that the control is the product of a *gain matrix* G with the modal state, $\mathbf{u} = G\mathbf{y}$.

As a very important special case, consider a situation where only one Lyapunov exponent λ_1 is greater than zero, and the control $\mathbf{u}$ is a scalar. Then we may take $G = g$ a scalar, while $\mathcal{E}^{-1}B = \mathbf{c}(t)$, a vector function of time. [To produce this function, it is not necessary to invert $\mathcal{E}$ at each time step. Differentiating $\mathcal{E}^{-1}\mathcal{E} = I$ and substituting from (15) produces

$$\frac{d}{dt}\mathcal{E}^{-1} = -\mathcal{E}^{-1}A + J\mathcal{E}^{-1}. \tag{23}$$

Since $\mathcal{E}(t_0)$ is an orthonormal matrix, its inverse is its transpose at $t = t_0$.] To stabilize the trajectory against small displacements, the positive Lyapunov exponent must be changed into a negative one, without altering the stability characteristics of the other exponents. But under these conditions, the system (22) becomes

$$\dot{\mathbf{y}} = \left\{\begin{array}{ccccc} \sigma_1 + c_1 g & 0 & 0 & \ldots & 0 \\ c_2 g & \sigma_2 & 0 & \ldots & 0 \\ c_3 g & 0 & \sigma_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_N g & 0 & 0 & \ldots & \sigma_N \end{array}\right\}\mathbf{y}. \tag{24}$$

Consider first the unstable mode y_1, which now has an equation of motion

$$\dot{y}_1 = \left[\sigma_1(t) + c_1(t)g(t)\right]y_1. \tag{25}$$

Since c_1 is certainly a function of time, and the gain g could be, this has solution

$$y_1(t) = y_1(t_0)\exp\left\{\int_{t_0}^{t}\left[\sigma_1(\tau) + c_1(\tau)g(\tau)\right]d\tau\right\}. \tag{26}$$

Now, write $c_1 g = \langle c_1 g\rangle + \Delta(t)$, where $\langle c_1 g\rangle$ is the average over the entire time interval

$$\langle c_1 g\rangle = \frac{1}{t_f - t_0}\int_{t_0}^{t_f} c_1(\tau)g(\tau)d\tau. \tag{27}$$

Similarly, write $\sigma_1(t) = \lambda_1 + \epsilon_1(t)$, where $\epsilon_1(t)$ represents the (not necessarily small) deviation of the instantaneous exponential rate σ_1 from its average value over the time interval. Then the solution for y_1, (26), becomes

$$\begin{aligned} y_1(t) = y_1(t_0)\exp\left\{(\lambda_1 + \langle c_1 g\rangle)(t - t_0)\right\} \\ \times\exp\left\{\int_{t_0}^{t}\left[\epsilon_1(\tau) + \Delta(\tau)\right]d\tau\right\}. \end{aligned} \tag{28}$$

Now, the integral of $\epsilon_1(t) + \Delta(t)$ over the interval $t_0 \le t \le t_f$ is zero, so it is clear that the new regional Lyapunov exponent for this interval, for the "closed loop state," is just

$$\lambda_1' = \lambda_1 + \langle c_1 g\rangle. \tag{29}$$

If we chose to use a constant feedback gain g, this can be solved for the gain as a function of the desired new exponent

$$g = (\lambda_1' - \lambda_1)/\langle c_1\rangle. \tag{30}$$

In control theory, such a relationship is termed a control law. If the average value $\langle c_1\rangle \ne 0$ (this is termed a *controllability condition*) then a gain g exists which will give any desired closed loop Lyapunov exponent. If $\langle c_1\rangle = 0$, one may still attempt to find a $g(t)$ which produces a nonzero value in (27).

Now, the closed loop equations of motion for the other states become

$$\dot{y}_i = \sigma_i(t)y_i + c_i(t)gy_1(t). \tag{31}$$

Regarding y_1 as a known function of time, and exponentially decaying assuming that we have chosen λ_1' to be negative, this is a variable coefficient linear equation with a time dependent forcing function. Its solution is found by elementary methods as

$$\begin{aligned} y_i(t) = y_i(t_0)\exp\left\{\int_{t_0}^{t}\sigma_i(\tau)d\tau\right\} \\ + \int_{t_0}^{t}\exp\left\{\int_{\tau}^{t}\sigma_i(\phi)d\phi\right\}c_i(\tau)gy_1(\tau)d\tau. \end{aligned} \tag{32}$$

By inspection, the homogeneous part is exponentially changing with the original exponential rate. The convolution integral above we will refer to as the zero state solution $y_{i,\mathrm{ZS}}(t)$, since it is the solution to (31) with zero initial condition.

In the case of time-periodic systems, Calico and Wiesel, this was sufficient to ensure complete pole placement. But in the general case, the presence of the zero state portion of the solution may change the other Lyapunov exponents. Figures 2 and 3 were computed for the Lorenz system with $t_f = 0.25$ and $t_f = 0.5$, respectively, and show the closed loop Lyapunov exponents as a function of gain g. In the first figure, the uncontrolled exponents are about $\lambda_i = 8.897, \ -2.1762, \ -27.721$. Some of these are not especially close to their $t_f = \infty$ values. The Lyapunov exponents for the closed loop system were calculated from

$$\dot{\mathbf{x}} = \left[A + Bg\left(\mathcal{E}^{-1}\right)_1\right]\mathbf{x}, \tag{33}$$

where $(\mathcal{E}^{-1})_1$ is the first row of $\mathcal{E}^{-1}$. This form of the closed loop problem is found by converting back to the physical variables. This is a time dependent linear system, and its Lyapunov exponents can be extracted

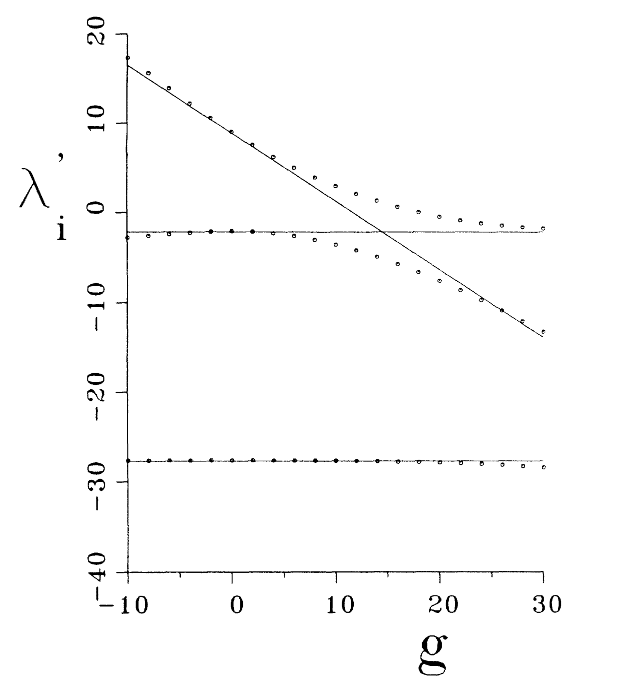

FIG. 2. Constant gain feedback for a final time $t_f = 0.25$. The Lyapunov exponents fairly closely follow the predicted straight lines, except near the root crossover.

through the techniques in the preceding section. Constant gain control does fairly well in the region plotted, but in the region where λ_1 and λ_2 cross, the roots diverge from the desired straight line dependence. A more extreme case is given in Fig. 3, where the hoped for linear relation is obeyed for only very small gain values.

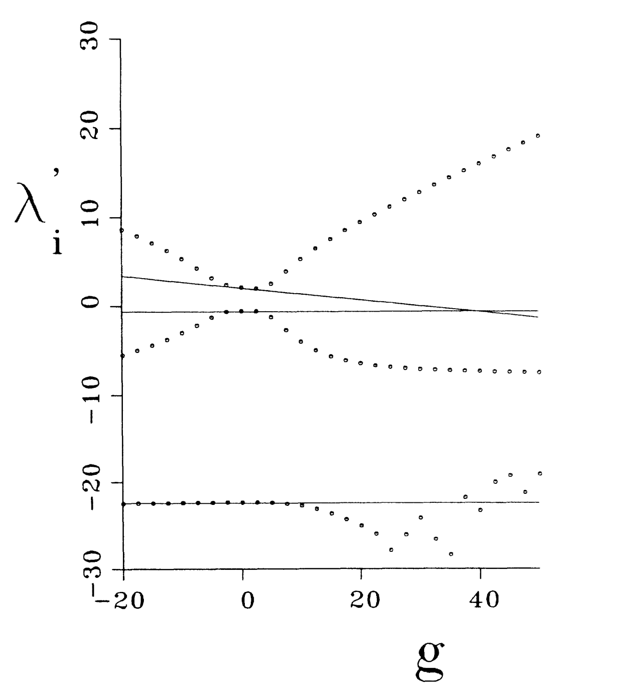

FIG. 3. Constant gain feedback for a final time $t_f = 0.5$ in the Lorenz system. Only very small gains follow the hoped-for linear relation.

There is no contradiction with Eq. (30), however, since that relation refers to the exponential growth rate along the *original* $\mathbf{e}_i(t)$ vectors. The presence of the zero state solutions for large enough gain values totally changes the orientation of the Lyapunov ellipsoid, and the closed loop Lyapunov exponents are quite different from the desired values.

This is a disappointment, since in the periodic coefficient case a constant gain is virtually always sufficient to ensure pole placement. However, there is still considerable freedom at our disposal if we consider the gain to be a function of time. All the results of this section [with the exception of (30)] are still correct in this case. For ease of calculation, consider writing the gain as a polynomial in time

$$g(t) = \sum_{i=0}^{N-1} g_i \,(t - t_0)^i \,, \tag{34}$$

which is, of course, also a Taylor's series. We propose to determine the coefficients by imposing the conditions that

$$\frac{1}{t_f - t_0} \int_{t_0}^{t_f} c_1(\tau) g(\tau) d\tau = \lambda_1' - \lambda_1, \tag{35}$$

and that the solution to (31) with $y_i(t_0) = 0$, the zero state solutions $y_{i,\mathrm{ZS}}$, should be zero at the final time. This reduces (32) at the final time to just

$$\begin{aligned} y_i(t_f) &= y_i(t_0) \exp\left\{ \int_{t_0}^{t_f} \sigma_i(\tau) d\tau \right\} \\ &= y_i(t_0) \exp\left\{ \lambda_i (t_f - t_0) \right\}. \end{aligned} \tag{36}$$

In other words, to ensure that only λ_1 is changed, the zero state solutions must vanish at the final time. The remaining homogeneous solution is then just the original uncontrolled response, at least at the end of the interval.

To meet these N conditions in an Nth order system, we only need a polynomial of degree $N - 1$. The coefficients have been calculated by a gradient method, and convergence was found to be quite rapid. The program simultaneously integrates the Lorenz system, the $\mathcal{E}$ matrix, the three separation conditions, and their linearization matrix. Results are shown in Figs. 4 and 5 as a function of the desired shift in the first Lyapunov exponent, $\lambda_1 - \lambda_1'$. It is apparent that this method is successful in moving λ_1 while leaving the other Lyapunov exponents unchanged. What is not apparent from the figures is how successful: in double precision arithmetic, the desired exponents were usually obtained to seven or eight significant figures. As earlier, the closed loop system response was calculated by integrating the fundamental matrix Φ for the closed loop linear system (33), with the calculated gain function over the interval, and obtaining the eigenvalues of $\Phi^T \Phi$.

Our pole placement technique is quite special, in that the system is left in diagonalized form at $t = t_f$. This means that the modal matrix for the closed loop system $\mathcal{E}'$ is the same as the modal matrix for the original system *at the initial and final times* . The separation conditions imposed to find the gain function $g(t)$ force this to be true

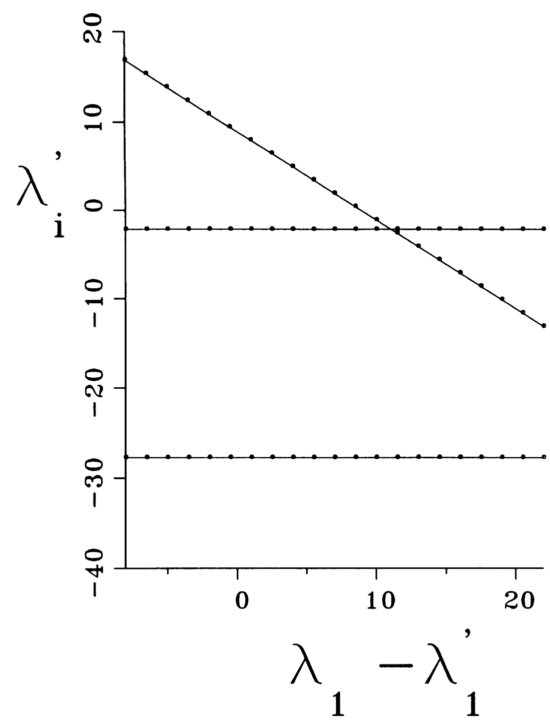

FIG. 4. Polynomial gain feedback with a final time $t_f = 0.25$. The closed loop system Lyapunov exponents have the expected form: the controlled root moves, while the others remain in their original positions.

at the final time, while it is trivially true at the initial time, since the control has yet to be applied at the initial instant. Figure 6 shows the closed loop Lorenz system for a final time $t_f = 0.5$, with a gain function moving the formerly unstable Lyapunov exponent to $\lambda_1' = -2.9511$, while leaving the other roots at -0.6365, -22.4123, respectively. It is shown projected onto the $\mathbf{z}$ axes, the modal axes for the closed loop system. Again, the plot is limited to $|z_i| \leq 0.001$, and z_1 is the formerly unstable axis. Trajectories converge along all three directions towards the origin (e.g., the reference trajectory). Some oscillation along the z_1 axis is apparent, but after a slight outward movement all trajectories reverse direction and approach the origin. This emphasizes that we have *not* produced a constant coefficient separated system (18), but a time dependent modal system. It is only the final, average exponential rates which have been prescribed.

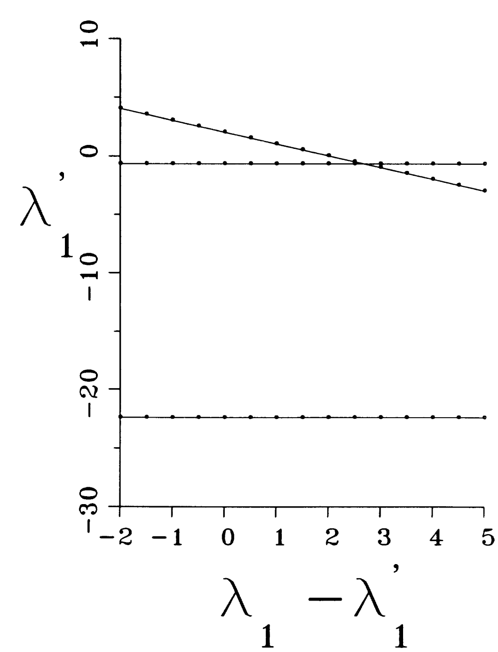

FIG. 5. Polynomial gain feedback for a final time $t_f = 0.5$. Again, the expected linear behavior occurs.

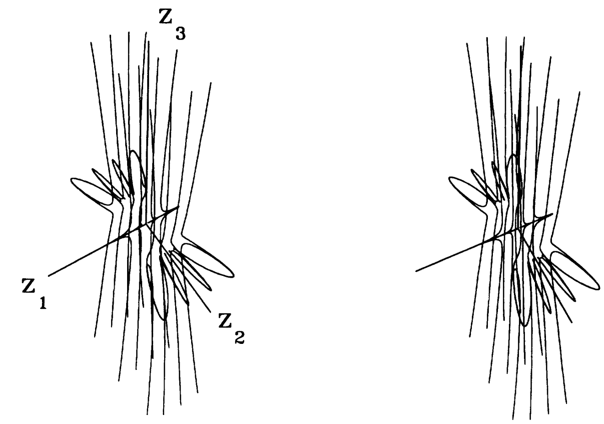

FIG. 6. Stereo plot of trajectories on the tangent space, projected along the modal vectors $\mathbf{e}_i'$ of the closed loop system. All three axes are now attracting, although some oscillations are apparent.

We have discussed the problem of changing one Lyapunov exponent while leaving the others unchanged. This is not as special a case as it may appear at first glance. With a gain chosen to move λ_1, Eq. (33) becomes a time dependent linear system of the original form (2). Once the first Lyapunov exponent is adjusted, the closed loop system can be rediagonalized, introducing new modal variables $\mathbf{z}$. Rediagonalization is necessary since the old $\mathbf{e}_i$ vectors only diagonalize the closed loop system at the initial and final times. But, as discussed above, the new and old modal matrices agree at the initial and final times. Since it is a linear system, additional control terms can be added to the system, and another Lyapunov exponent can be selected for control. This technique has been applied with fair success to several time-periodic problems, e.g., Webb, Calico, and Wiesel [15]. While there is no theoretical limit to the number of Lyapunov exponents which can be controlled in this fashion, we have noted a slow loss of significant figures when controlling multiple modes in time-periodic systems. It is expected that this will be the case for the general case solved in this paper.

It is not necessary to use a polynomial to express the gain function $g(t)$, although this is also a Taylor's series, and the times we have been able to handle in the Lorenz system are quite short. Other basis functions may suggest themselves in other problems. By retaining additional terms in the Taylor's series, the freedom may be gained to do optimal control. For example, one may minimize the mean square control amplitude

$$\langle g(t)\rangle^2 = \frac{1}{t_f - t_0}\int_{t_0}^{t_f} g^2(\tau)d\tau, \quad (37)$$

subject to the separation conditions as constraints:

$$\langle c_1 g\rangle = \lambda_1' - \lambda_1, \quad y_{i,\mathrm{ZS}}(t_f) = 0. \quad (38)$$

This type of problem is usually solved with Lagrange multipliers, as in Sec. II. The definition of what is "optimal" will vary greatly from problem to problem, however. For example, minimizing

$$|g|^+ = \sup_{t_0 \le t \le t_f} |g(t)| \quad (39)$$

on the interval $(t_0,\ t_f)$ would produce a minimum amplitude control in the Chebyshev sense, rather than the least squares sense. We intend to report on some further examples shortly.

V. DISCUSSION AND CONCLUSIONS

In this paper it has been demonstrated that the tangent space of any dynamical system can be decoupled into "normal modes," locally noninteracting subspaces. Solutions on each modal vector expand or contract exponentially with a Lyapunov exponent particular to that mode. A pole placement technique which changes one Lyapunov exponent by a prescribed amount, while leaving all others at their original values, has been derived. Iteration of this technique should make it possible to deterministically change more than one Lyapunov exponent.

The technique is limited to finite arcs of trajectory, because the regional Lyapunov exponents are obtained from the eigenvalues of $\Phi^T\Phi$. However, longer periods of time could be handled by simply analyzing another arc beginning where the first arc ends, and changing variables at the splicing time. If each arc is long enough to let the regional Lyapunov exponents approach their thermodynamic values, the mismatch in the modal matrices $\mathcal{E}$ should not be too severe.

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